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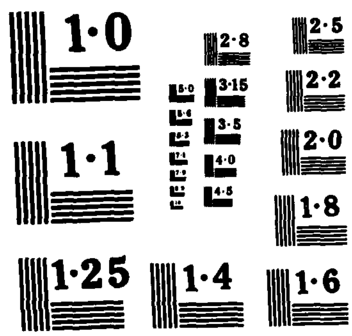
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MRC Technical Summary Report #2824

A FLOQUET DECOMPOSITION FOR VOLTERRA EQUATIONS WITH PERIODIC KERNEL AND A TRANSFORM APPROACH TO LINEAR RECURSION EQUATIONS

David L. Russell

MIPAC Facility Document No. 2

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

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ABSTRACT

We show that a functional equation in E^n:

$$x(t) = \int_{-\infty}^{t} W(t,s)x(s)ds$$

with piecewise continuous $m \times m$ matrix kernel $W(t,s)$ satisfying, for $T,\ \Omega,\ \mu > 0$

$$W(t + T,\ s + T) = W(t,s)\ ,$$

$$\|W(t,s)\| < \Omega e^{-\mu(t-s)}, t, s \text{ real}$$

admits, for each $\beta < \mu$, a decomposition, applicable to a wide class of solutions $x(t)$ for $t > 0$,

$$x(t) = x_F(t) + x_\beta(t)$$

where, for some $B = B(\beta)$,

$$\|x_\beta(t)\| < Be^{-\beta t},\quad t > 0$$

and $x_F(t)$ is a linear combination of "Floquet type" solutions

$$t^q e^{\lambda t} p(t), q(> 0) \in Z,\ \lambda \in C,\ Re(\lambda) > -\beta\ ,$$

$p(t)$ being a continuous n-vector function such that

$$p(t + T) = p(t)\ .$$

*Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706. This research was completed while the author was visiting at the University of Florida, Gainesville.

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The theorem is proved by converting the above equations to a convolution type linear recursion equation

$$\sum_{k=-\infty}^{0} Q_k x_{k+j} = 0$$

in $L_m^2[0,T]$ and studying this equation by transform methods. In the process we examine some general properties of equations

$$\sum_{k=-\infty}^{\infty} Q_k x_{k+j} = 0$$

within the same transform framework.

AMS (MOS) Subject Classifications: 39A10, 39A11, 39A12, 44A10, 44A50, 34K20, 34K30

Key Words: Floquet theory, linear recursion equations, difference equations, periodic systems, delay systems, integral equations

Work Unit Number 1 (Applied Analysis)

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SIGNIFICANCE AND EXPLANATION

This report provides the theoretical background for ~~our~~ an earlier report a ~~(TSR #2771,~~ ("Frequency/Period Estimation and Adaptive Rejection of Sinusoidal Disturbances"). It also lays the foundation for a new type of transform analysis for linear recursion equations which appears to be quite useful in analyzing their spectral properties. Additional keywords: Floquet theory; difference equations; integral equations; linear control systems.

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A FLOQUET DECOMPOSITION FOR VOLTERRA EQUATIONS WITH PERIODIC KERNEL AND A TRANSFORM APPROACH TO LINEAR RECURSION EQUATIONS

David L. Russell*

MIPAC Facility Document No. 2

1. STATEMENT OF PRINCIPAL RESULTS

The present work developed, originally, in a supporting role in connection with the stability of certain adaptive frequency rejection procedures for linear control systems ([6]), where small variations in the estimate, $\hat{T}(t)$, of the period, T, of the incoming disturbance, were seen to satisfy an equation of the type appearing in Theorem 1, below, with $m = 1$. The specific question of interest concerned whether or not the asymptotic stability of such a system can be decided on the basis of knowledge of "Floquet type" solutions

$$t^q e^{\lambda t} p(t)$$

of the system, where q is a non-negative integer, λ a complex number, and $p(t)$ a continuous T-periodic m-vector function. This is an important question for applications because such solutions are the easiest to identify by computational procedures. The question is answered in the affirmative by

Theorem 1. *Consider the vector functional equation in* E^m:

$$x(t) = \int_{-\infty}^{t} W(t,s)x(s)ds \tag{1.2}$$

where $W(t,s)$ *is a piecewise continuous* $m \times m$ *matrix function satisfying*

$$\|W(t,s)\| \le \Omega e^{-\mu(t-s)}, \quad -\infty < s \le t, \tag{1.3}$$

for positive numbers Ω, μ *and is periodic in the sense that*

$$W(t + T, s + T) = W(t,s) \tag{1.4}$$

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for a certain positive minimal period T. Then, given any real $\nu < \mu$, a solution $x(t)$ of (1.2) corresponding to an initial "history"

$$x(-s), \quad 0 \le s < \infty ,$$

with

$$\|x(-s)\|_{E^m} \le \Xi e^{\nu s}, \quad 0 \le s < \infty \tag{1.5}$$

for some positive Ξ and real $\nu < \mu$ (which may depend on x) can be written as

$$x(t) = x_F(t) + x_\beta(t) , \tag{1.6}$$

where $x_F(t)$ is a finite linear combination of Floquet type solutions with $\mathrm{Re}(\lambda) > -\beta$ and

$$\|x_\beta(t)\| \le B e^{-\beta t}, \quad t \ge 0 \tag{1.7}$$

where B is a positive number:

$$B = B_0(\beta)\left(\int_0^\infty \|x(-s)\|^2 ds\right)^{1/2} .$$

In particular, since β may be taken to be positive,

$$\lim_{t\to\infty} \|x(t)\| = 0$$

for all solutions $x(t)$ as described if and only if $\mathrm{Re}(\lambda) < 0$ for all Floquet type solutions $t^q e^{\lambda t} p(t)$.

Remarks. The condition (1.5), or something similar, is necessary to ensure the convergence of the integral in (1.2). It is trivially satisfied, of course, if $x(-s) \equiv 0$ for $s > s_0$ for some positive s_0, normally the case in applications.

Theorem 1 is proved as an application of another result, Theorem 2 below, which concerns linear recursion equations of convolution type, i.e., with constant coefficients. To set the stage for this result, let X be a Banach space with norm $\| \, \|$ and

$$x = \{x_k \mid -\infty < k < \infty\}$$

a sequence of vectors in X having exponentially bounded growth as k tends to $-\infty$, i.e., there exist positive numbers M, γ

$$\|x_k\| \leq M(\gamma)^k, \quad k = 0,-1,-2,-3,\ldots . \tag{1.8}$$

Suppose further that Q_k, $k = 0,-1,-2,-3,\ldots$ are bounded operators on X such that

$$\sum_{k=-\infty}^{0} \|Q_k\|\rho^k < \infty \tag{1.9}$$

for some ρ satisfying $0 < \rho < \gamma$. Defining the bounded operator valued functions $Q(z)$ by

$$Q(z) \equiv \sum_{k=-\infty}^{0} Q_k z^k \tag{1.10}$$

we see that $Q(z)$ is an analytic operator valued function for $|z| > \rho$. Equally well

$$\xi^-(z) \equiv \sum_{k=-\infty}^{0} x_k z^{-k} \tag{1.11}$$

is an analytic X-vector valued function for $|z| < \gamma$. We let D be a circle of radius d centered at 0 with $\gamma > d > \rho$ and we let C be a similar circle of radius $c > d$, both circles oriented positively. Using certain results about the "transforms" $Q(z)$, $\xi^-(z)$, which we will develop in Sections (2) and (3), we are able to prove

Theorem 2. Let $\{x_k | -\infty < k < \infty\}$ satisfy the convolution equation

$$\sum_{k=-\infty}^{0} Q_k x_{k+j} = 0, \quad j = 1,2,3,\ldots \tag{1.12}$$

with $x_k \in X$ satisfying (1.8) for $k \leq 0$. Let Q_0 be nonsingular and assume that $Q(z)$ is "regular" in the sense that its singular points ζ, points such that $Q(\zeta)^{-1}$ does not exist as a bounded operator on X, have no cluster points in $|\zeta| > \rho$ and are such that the null space of $Q(\zeta)$ is finite dimensional in each case. Let D, as described above, be situated so as not to pass through any singular point ζ of $Q(z)$ and let C be selected so that $c > |\zeta|$ for all singular points ζ of $Q(z)$. Then, given any σ, $c > \sigma > \rho$, and S a positively oriented circle, centered at 0, of radius σ, we may assume without loss of generality that $d < \sigma$ and we have, for $k = 1,2,3,\ldots$

$$x_k = \frac{1}{2\pi i}\int_{C-S} z^{k-1}Q(z)^{-1}q(z)dz + \frac{1}{2\pi i}\int_S z^{k-1}Q(z)^{-1}q(z)dz \equiv x_{k,F} + x_{k,\sigma}\,, \quad (1.13)$$

where

$$q(z) = \frac{1}{2\pi i}\int_D \frac{Q(\zeta)\xi^{-}(\zeta)d\zeta}{z-\zeta}, \quad |z| > d\,. \quad (1.14)$$

As a consequence we have, for some $N = N(\sigma) > 0$,

$$\|x_{k,\sigma}\| \le N\sigma^k, \quad k = 1,2,3,\ldots \quad (1.15)$$

and

$$x_{k,F} = \sum_{\zeta \in Z(C,S)} \mathrm{Res}(z^{k-1}Q(z)^{-1}q(z)|_{z=\zeta}\,, \quad (1.16)$$

where $Z(C,S)$ is the set of singular points of $Q(z)$ between C and S. If the dimension of the null space of $Q(\zeta)$ is ν_ζ, then $\mathrm{Res}(z^{k-1}Q(z)^{-1}q(z))|_{z=\zeta}$ is a linear combination of solutions of (1.12) having the form

$$k^{\mu}\zeta^{k}p, \quad k = 1,2,3,\ldots \quad (1.17)$$

where μ is an integer, $0 \le \mu < \nu_\zeta$, and p is a non-zero vector in X which is a generalized eigenvector in the sense that

$$Q^{(m)}(\zeta)p = 0, \quad m = 0,\ldots\mu\,. \quad (1.18)$$

In particular, if $\rho < 1$ and σ is selected so that $1 > \sigma > \rho$, then

$$\lim_{k\to\infty} \|x_k\| = 0$$

for all solutions $\{x_k\}$ of (1.12) satisfying (1.8) just in case all of the singular points ζ of $Q(z)$ satisfy $|\zeta| < 1$.

Remark. A theorem of F. V. Atkinson (see the original paper [1] or the treatment by T. Kato in [4]) shows that $Q(z)$, with Q_0 nonsingular, is regular, as defined, if the Q_k are compact $k = -1,-2,-3,\ldots$.

In Section 2 we develop the machinery necessary to prove Theorem 2, but in a slightly more general setting (allowing equations of the form $\sum_{k=-\infty}^{\infty} Q_k x_{k+j} = 0$). In Section 3 we prove Theorem 2 and some other results for linear recursion equations of "unilateral" type. Theorem 1 is proved in Section 4.

Theorem 1 extends to a class of functional equations involving an infinite delay results already presented in [3] for certain equations with finite delay, namely those of neutral and retarded type. It is likely that a modification of the methods used here would provide alternative proofs of those theorems.

2. A TRANSFORM THEORY FOR LINEAR RECURSION EQUATIONS

Here, as indicated earlier, we take a somewhat more general point of view than what is minimally required in order to prove Theorem 2. Let X be a Banach space as in Section 1 and let us consider sequences

$$x = \{x_k \mid -\infty < k < \infty\}$$

of vectors in X having exponentially bounded growth as k tends to infinity in both directions; there are positive numbers M^+, M^-, γ^+, γ^-, with $\gamma^+ > \gamma^-$, such that

$$\|x_k\| \leq M^+(\gamma^+)^k, \quad k = 1,2,3,\ldots \tag{2.1}$$

$$\|x_k\| \leq M^-(\gamma^-)^k, \quad k = 0,-1,-2,-3,\ldots \; . \tag{2.2}$$

Along with such sequences of vectors we consider sequences

$$Q = \{Q_k \mid -\infty < k < \infty\} \tag{2.3}$$

of bounded operators $Q_k : X \to X$ which satisfy inequalities

$$\left.\begin{aligned} &\sum_{k=1}^{\infty} \|Q_k\|(\rho^+)^k < \infty , \\ &\sum_{k=-\infty}^{0} \|Q_k\|(\rho^-)^k \quad \left(= \sum_{k=0}^{\infty} \|Q_{-k}\| \frac{1}{(\rho^-)^k}\right) < \infty \end{aligned}\right\} , \tag{2.4}$$

where ρ^+, ρ^- are positive numbers with

$$\rho^+ > \gamma^+ > \gamma^- > \rho^- > 0 \; . \tag{2.5}$$

From this it is clear that the convolution product defined by

$$(Q * x)_\ell \equiv \sum_{k=-\infty}^{\infty} Q_k x_{k+\ell}$$

is convergent for every integer ℓ and we may consider the equation

$$Q * x = f \tag{2.6}$$

where

$$f = \{f_k \mid -\infty < k < \infty\}$$

is also a sequence in X with certain properties to be discussed subsequently.

In agreement with standard usage we define the "z-transform", or discrete Laplace transform, of x by

$$\xi(x,z) = \begin{cases} \sum_{k=1}^{\infty} x_k z^{-k} \quad (\equiv \xi^+(x,z)), & |z| > \gamma^+ , \\ -\sum_{k=0}^{\infty} x_{-k} z^k \quad (\equiv \xi^-(x,z)), & |z| < \gamma^- , \end{cases} \tag{2.7}$$

When $x = \{x_k\}$ is clear from the context we will simply write $\xi(z)$. Clearly $\psi(z)$ is analytic for $|z| > \gamma^+$ and for $|z| < \gamma^-$. In certain instances $\xi(z)$, as defined in one of these regions, is an analytic continuation of $\xi(z)$ as defined in the other. The most basic example, for $X = E^1$, concerns the sequence

$$x_k = \lambda^{k-1}, \quad -\infty < k < \infty ,$$

where λ is a non-zero complex number. Here

$$\gamma^+ = \gamma^- = |\lambda|$$

and we see that for $|z| > |\lambda|$

$$\xi(z) = \frac{1}{z} + \frac{\lambda}{z^2} + \frac{\lambda^2}{z^3} + \dots = \frac{1}{z}\left(\frac{1}{1 - \lambda/z}\right) = \frac{1}{z - \lambda}$$

while for $|z| < |\lambda|$

$$\xi(z) = -\frac{1}{\lambda} - \frac{z}{\lambda^2} - \frac{z^2}{\lambda^3} - \dots = -\frac{1}{\lambda}\left(\frac{1}{1 - z/\lambda}\right) = \frac{1}{z - \lambda} .$$

We will see as we proceed that those cases wherein $\xi^+(z)$ and $\xi^-(z)$, as defined by (2.7), are analytic continuations of each other, correspond, if $Q * x = 0$, to rather particular solutions of that equation.

The z-transform is readily inverted by taking C^+ and C^- to be circles with positive radii $c^+ > \gamma^+$, $c^- < \gamma^-$, centered at 0, positively oriented, and verifying that

$$x_k = \begin{cases} \frac{1}{2\pi i} \int_{C^+} \xi(z) z^{k-1} dz, & k = 1,2,3,\ldots , \\ -\frac{1}{2\pi i} \int_{C^-} \frac{\xi(z)dz}{z^{k+1}}, & k = 0,-1,-2,\ldots . \end{cases} \tag{2.8}$$

If we set

$$C = C^+ - C^-$$

and use Cauchy's theorem, we see that

$$x_k = \frac{1}{2\pi i} \int_C \xi(z) z^{k-1} dz, \quad -\infty < k < \infty . \tag{2.9}$$

For Q as in (2.3)-(2.5) we define a variety of "discrete Fourier transform"

$$Q(z) = \sum_{k=-\infty}^{\infty} Q_k z^k ,$$

analytic for $\rho^- < |z| < \rho^+$. There exists, as one would expect, an important relationship between $Q(z)$, $\xi(z)$ and the z-transform $\phi(z)$ of f when Q, x and f satisfy (2.6). In order to explain this relationship we need to introduce a certain decomposition which we will call the "internal-external" decomposition.

Let $D = D^+ - D^-$ be a contour similar to C described above, D^+ and D^- having radii d^+ and $d^-(<d^+)$, respectively. Let h be a complex-valued function defined and square integrable on D. Let

$$\hat{h}(z) = \frac{1}{2\pi i} \int_D \frac{h(\zeta)d\zeta}{z - \zeta}, \quad |z| > d^+, \ |z| < d^- , \tag{2.10}$$

$$\tilde{h}(z) = \frac{1}{2\pi i} \int_D \frac{h(\zeta)d\zeta}{\zeta - z}, \quad d^- < |z| < d^+ , \tag{2.11}$$

which we call the exterior and interior functions (relative to the particular contour C) associated with h, respectively analytic in the exterior of D and the interior of D. From familiar results for Fourier series it is easy to see that if $\hat{D}_\varepsilon$, $\tilde{D}_\varepsilon$ are contours of the same sort as D, lying a distance ε in the exterior, interior, respectively, of

$$\phi^+(z) = \sum_{k=1}^{\infty} f_k z^{-k}$$

be holomorphic for $|z| > \gamma^+$. Assuming

$$Q(z) = \sum_{k=-\infty}^{0} Q_k z^k ,$$

Q_0 nonsingular, is holomorphic for $|z| > \rho^-$, $0 < \rho^- < \gamma^-$, the z-transform $\xi(z)$ of the unilateral solution x of $Q * x = f$ with initial history x^- is such that the transform of

$$x^+ = \{x_k | 1 \le k < \infty\}$$

is given by

$$\xi^+(z) = \frac{1}{2\pi i} \int_{D^+} \frac{Q(\zeta)^{-1}\phi^+(\zeta)d\zeta}{z - \zeta} + \frac{1}{2\pi i} Q(z)^{-1} \int_{D^-} \frac{Q(\zeta)\xi^-(\zeta)d\zeta}{z - \zeta} \tag{3.2}$$

where D^+ and D^- with radii d^+ and d^- are selected so that $d^+ > \gamma^+$ and $Q(z)$ is nonsingular for $|z| > d^+$ while $\rho^- < d^- < \gamma^-$, and the formula (2.23) is valid for $|z| > d^+$.

Proof. We first construct the solution in the unilaterally homogeneous case wherein $f_k = 0$, $k = 1,2,3,\ldots$. Let D^- have radius d^- as indicated and let us define

$$\xi_0^+(z) = \frac{1}{2\pi i} Q(z)^{-1} \int_{D^-} \frac{Q(\zeta)\xi^-(\zeta)d\zeta}{z - \zeta} .$$

Since Q_0 is nonsingular, there is a positive number r such that $Q(z)$ is nonsingular for $|z| > r$. Let $C = C^+ - C^-$ be such that $c^+ > r$ and $\gamma^- > c^- > \rho^-$. Let

$$\xi_0(z) = \begin{cases} \xi_0^+(z), & |z| > \gamma^+ \\ \xi^-(z), & |z| < \gamma^- \end{cases}$$

and let us compute, for w exterior to C,

3. UNILATERAL EQUATIONS AND THE PROOF OF THEOREM 2.

Unilateral equations are equations of the form $Q * x = f$ wherein

$$Q = \{Q_k | -\infty < k < \infty\}$$

has the additional property that

$$Q_k = 0, \quad k > 0 ,$$

and thus the equation takes the form

$$\sum_{k=-\infty}^{0} Q_k x_{k+j} = f_j . \tag{3.1}$$

We will say that x is a unilateral solution if this equation is valid for $j = 1,2,3,\ldots$ (otherwise x is a complete solution, if the distinction needs to be made; up to the present we have referred to complete solutions merely as solutions). We will say that x is a unilateral homogeneous solution, or just that it is unilaterally homogeneous, if $f_j = 0$ for $j = 1,2,3,\ldots$.

For unilateral equations the natural problem is the initial history problem wherein we assume that for $k = 0,-1,-2,-3,\ldots x_k$ is given *a priori*. Ordinarily we accept any such sequence, call it x^-, for which $\xi^-(z)$ (cf. (2.7)) is holomorphic for $|z| < \gamma^-$, without being concerned about the values of $(Q * x)_j$ for $j \le 0$. The sequence x is continued for $k = 1,2,3,\ldots$ by enforcing (3.1), for a given f^+, i.e., $\{f_j | j = 1,2,3,\ldots\}$, for $j > 0$; to ensure this continuability we assume that Q_0 is nonsingular. The problem, then, is to characterize the continuation sequence

$$x^+ = \{x_k | k = 1,2,3,\ldots\}$$

in terms of x^-, f^+ and Q. The following theorem does just that, in terms of the transforms of those sequences.

Theorem 9. *Let $x^- = \{x_k | -\infty < k \le 0\}$ with z transform*

$$\xi^-(z) = \sum_{k=-\infty}^{0} x_k z^{-k}$$

be holomorphic for $|z| < \gamma^-$ and let points of $Q(z)$ between the two contours. Thus, as we should expect, there is an intrinsic relationship between solutions of $Q * x = 0$ and singular points of $Q(z)$.

In order to distinguish the concept of homogeneous solutions as developed here from the more restricted one to be developed presently, we will refer to solutions x of $Q * x = 0$, which means $(Q * x)_j = 0$ for $-\infty < j < \infty$ or equivalently, that

$$\widehat{Q(z)\xi(z)} = 0 \tag{2.33}$$

for the z-transform, ξ, of x, as *completely homogeneous* solutions. If equation (2.33) is true for an appropriate contour D, for z on that contour

$$Q(z)\xi(z) = \widetilde{Q(z)\xi(z)} + \widehat{Q(z)\xi(z)} = \widetilde{Q(z)\xi(z)} .$$

Now, for z in the interior of D,

$$\widetilde{Q(z)\xi(z)} = \frac{1}{2\pi i}\int_D \frac{Q(\zeta)\xi(\zeta)d\zeta}{\zeta - z}$$

and thus $Q(z)\xi(z)$, initially defined only for $\rho^- < |z| < \gamma^-$ and $\gamma^+ < |z| < \rho^+$, which includes D, must, as a consequence of the identity theorem, have the analytic continuation

$$Q(z)\xi(z) = \frac{1}{2\pi i}\int_D \frac{Q(\zeta)\xi(\zeta)d\zeta}{\zeta - z}$$

throughout the interior of D and thus, in fact, $Q(z)\xi(z)$ is holomorphic for $\rho^- < |z| < \rho^+$. Since $\xi(z)$ is holomorphic for $|z| < \gamma^-$ and for $|z| > \gamma^+$, which intersects $\rho^- < |z| < \rho^+$ in two open annuli, we can summarize in

Theorem 8. *The sequence x is a completely homogeneous solutions, i.e., $(Q * x)_j = 0$ for $-\infty < j < \infty$, if and only if $\psi(z) \equiv Q(z)\xi(z)$ is extendable as a holomorphic function to $\rho^- < |z| < \rho^+$. If the set of singularities of $Q(z)$ has a connected complement, then (cf. (2.7)) $\xi^+(z)$ and $\xi^-(z)$, holomorphic, respectively, in $|z| > \gamma^+$ and $|z| < \gamma^-$, are each analytic continuations of the function $Q(z)^{-1}\psi(z)$ and hence analytic continuations of each other.*

for z exterior to D. Then for w between C and D, relative to C,

$$Q(w)\eta(w) = \frac{1}{2\pi i}\int_C \frac{Q(z)\eta(z)dz}{z-w} = -\frac{1}{4\pi^2}\int_C \frac{Q(z)}{z-w}\int_D \frac{Q(\zeta)^{-1}\theta(\zeta)d\zeta}{z-\zeta}\,dz$$

$$= -\frac{1}{4\pi^2}\int_D\int_C Q(z)\Big[\frac{1}{z-\zeta}-\frac{1}{z-w}\Big]dz\,\frac{Q(\zeta)^{-1}\theta(\zeta)}{\zeta-w}\,d\zeta$$

$$= \frac{1}{2\pi i}\int_D \frac{\theta(\zeta)}{\zeta-w}\,d\zeta + Q(w)\,\frac{1}{2\pi i}\int_D \frac{Q(\zeta)^{-1}\theta(\zeta)d\zeta}{w-\zeta} = 0 + Q(w)\eta(w) \qquad (2.32)$$

Since $Q(w)\eta(w)$ is holomorphic on C, we conclude that it is internal relative to C. Then, for z exterior to C,

$$(\tilde{\Pi}_C\tilde{\Pi}_D\xi)(z) = \frac{1}{2\pi i}\int_C \frac{Q(w)^{-1}}{z-w}\,\widehat{Q(w)\eta(w)}dw$$

$$= \text{(using (2.30), (2.31))} = \frac{1}{2\pi i}\int_C \frac{\eta(w)}{z-w}\,dw = \eta(z) = (\tilde{\Pi}_D\xi)(z)$$

since η, being external relative to D, is also external relative to C. This completes the proof of Proposition 7.

Letting C tend to D, we conclude that both $\tilde{\Pi}_D$ and $\hat{\Pi}_D$ are projections. We may think of $\tilde{\Pi}_D$ as extracting, in view of (Hom) in the proof of Theorem 5 (taking $\psi(z) = Q(z)\xi(z)$), a "homogeneous" part of $\xi(z)$ in the sense that

$$\widehat{Q(z)(\tilde{\Pi}_D\xi)(z)} = 0$$

and thus the sequence $\tilde{x}_D$ whose z-transform is $(\tilde{\Pi}_D\xi)(z)$ satisfies $Q * \tilde{x}_D = 0$. Correspondingly, $\hat{\Pi}_D$ extracts an "inhomogeneous" part, $(\hat{\Pi}_D\xi)(z)$, of $\xi(z)$, for which

$$\widehat{Q(z)(\hat{\Pi}_D\xi)(z)} = \phi(z)$$

and the sequence $\hat{x}_D$ whose z-transform is $(\hat{\Pi}_D\xi)(z)$ satisfies $Q * x = f$. Proposition 7 shows that, as the contour D expands, the homogeneous part grows larger in the sense that the range of $\tilde{\Pi}_C$ includes that $\tilde{\Pi}_D$ when C is exterior to D. Whether or not $\tilde{\Pi}_C$ and $\tilde{\Pi}_D$ are different, or agree, respectively, depends on whether or not there are singular

$$(\hat{\Pi}_D\xi)(w) = ((I - \tilde{\Pi}_D)\xi)(w) = \frac{1}{2\pi i}\int_D \frac{Q(z)^{-1}\phi(z)dz}{w - z} = \frac{1}{2\pi i}\int_D \frac{Q(z)^{-1}\widehat{Q(z)\xi(z)}dz}{w - z}$$

where $\widehat{Q(z)\xi(z)}$ is the external part of $Q(z)\xi(z)$ relative to any contour K with the properties described earlier, and, as noted earlier, $\widetilde{Q(z)\xi(z)}$ is the internal part of $Q(z)\xi(z)$ relative to C. When $Q(z)^{-1}$ is holomorphic on C and on K and on the annular regions between them, which includes D, both $\sim$ and $\hat{}$ can be interpreted relative to the contour D over which the integration is performed. We will assume this to be the case for purposes of defining $\tilde{\Pi}_D$ and $\hat{\Pi}_D$ as above. Thus $\tilde{\Pi}_D$ and $\hat{\Pi}_D$ are defined entirely with reference to the contour D.

Now let C, unrelated to C in the previous paragraph except that we continue to assume (2.16), (2.18) hold, be a contour exterior to D, still of the same general type; $C = C^+ - C^-$. Assuming no singularity of $Q(z)$ lies on C either, we can define $\tilde{\Pi}_C$ and $\hat{\Pi}_C$ for the contour C. Then we have

Proposition 7. *For w exterior to C*

$$(\tilde{\Pi}_C\tilde{\Pi}_D\xi)(w) = (\tilde{\Pi}_D\xi)(w) .$$

Remark. Then also, quite clearly

$$(\hat{\Pi}_C\hat{\Pi}_D\xi)(w) = (\hat{\Pi}_C\xi)(w) ,$$

for

$$((I - \tilde{\Pi}_C)(I - \tilde{\Pi}_D)\xi)(w) = ((I - \tilde{\Pi}_C - \tilde{\Pi}_D + \tilde{\Pi}_D)\xi)(w)$$

$$= ((I - \tilde{\Pi}_C)\xi)(w) = (\hat{\Pi}_C\xi)(w)$$

Proof of Proposition 7. Let

$$\theta(\zeta) = \widetilde{Q(\zeta)\xi(\zeta)} \tag{2.30}$$

relative to D. Then let

$$\eta(z) = \frac{1}{2\pi i}\int_D \frac{Q(\zeta)^{-1}\theta(\zeta)d\zeta}{z - \zeta} = (\tilde{\Pi}_D\xi)(z) \tag{2.31}$$

Let $K = K^+ - K^-$ be such that $\gamma^- < k^- < d^-$ and $d^+ > k^+ > \gamma^+$. Then

$$\frac{1}{2\pi i}\int_D \frac{Q(z)^{-1}\phi(z)dz}{w-z} = \frac{-1}{4\pi^2}\int_D \frac{Q(z)^{-1}}{w-z}\int_K \frac{Q(\zeta)\xi(\zeta)d\zeta}{z-\zeta}\,dz \tag{2.27}$$

Now the Cauchy formula gives, for z lying between C and K,

$$Q(z)\xi(z) = \frac{1}{2\pi i}\left(\int_C - \int_K\right)\frac{Q(\zeta)\xi(\zeta)}{\zeta - z}\,d\zeta \tag{2.28}$$

and then, for w exterior to D,

$$\frac{1}{2\pi i}\int_D \frac{Q(z)^{-1}\phi(z)dz}{w-z} = \frac{1}{2\pi i}\int_D \frac{Q(z)^{-1}}{w-z}\left[\frac{1}{2\pi i}\int_C \frac{Q(\zeta)\xi(\zeta)d\zeta}{\zeta - z} + Q(z)\xi(z)\right]dz\ , \tag{2.29}$$

where we have solved for $\int_K$ in (2.28) and changed $\zeta - z$ to $z - \zeta$ in the denominator of its integrand before substituting into (2.27) to get (2.29). Now, by definition,

$$\frac{1}{2\pi i}\int_D \frac{Q(z)^{-1}}{w-z}\,\frac{-1}{2\pi i}\int_C \frac{Q(\zeta)\xi(\zeta)d\zeta}{\zeta - z}\,dz = \frac{-1}{2\pi i}\int_D \frac{Q(z)^{-1}}{w-z}\,\widetilde{Q(z)\xi(z)}dz$$

where $\sim$ denotes here the internal *function relative* to C. Since w is exterior to D

$$\frac{1}{2\pi i}\int_D \frac{Q(z)^{-1}Q(z)\xi(z)dz}{w-z} = \xi(w)$$

and therefore

$$\xi(w) = \frac{1}{2\pi i}\int_D \frac{Q(z)^{-1}(\phi(z) + \widetilde{Q(z)\xi(z)})}{w-z}\,dz\ ,$$

as claimed, and the proof is complete.

Some interesting results devolve from the equation (2.23) which deserve a few paragraphs' notice. We define

$$(\tilde{\Pi}_D\xi)(w) = \frac{1}{2\pi i}\int_D \frac{Q(z)^{-1}\widetilde{Q(z)\xi(z)}dz}{w-z}$$

Now, using the formula (2.21) for $\xi(z)$, still for w in the interior of D,

$$Q(w)\xi(w) = -\frac{1}{4\pi^2}\int_C \frac{Q(z)}{z-w}\int_D \frac{Q(\zeta)^{-1}(\psi(\zeta)+\phi(\zeta))d\zeta}{z-\zeta}\,dz$$

$$= -\frac{1}{4\pi^2}\int_D\int_C Q(z)\left(\frac{1}{z-\zeta}-\frac{1}{z-w}\right)dz\,\frac{Q(\zeta)^{-1}(\psi(\zeta)+\phi(\zeta))}{z-w}\,d\zeta$$

$$= \frac{1}{2\pi i}\int_D \frac{(\psi(\zeta)+\phi(\zeta))d\zeta}{\zeta-w} - Q(w)\,\frac{1}{2\pi i}\int_D \frac{Q(\zeta)^{-1}(\psi(\zeta)+\phi(\zeta))d\zeta}{\zeta-w}\,. \qquad (2.26)$$

Combining (2.25) with (2.26) we have

$$Q(w)^{-1}(\psi(w) - \widetilde{Q(w)\xi(w)}) = \frac{1}{2\pi i}\int_D \frac{Q(\zeta)^{-1}(\psi(\zeta)+\phi(\zeta))d\zeta}{\zeta-w}$$

from which we conclude that (2.22) is internal relative to D, as required for the theorem. Then the formula (2.21) for $\xi(z)$ immediately gives the result (2.23), for, if $Q(\zeta)^{-1}(\psi(\zeta) - \widetilde{Q(\zeta)\xi(\zeta)})$ is internal relative to D,

$$\frac{1}{2\pi i}\int_D \frac{Q(\zeta)^{-1}(\psi(\zeta) - \widetilde{Q(z)\xi(\zeta)})d\zeta}{z-\zeta} = 0$$

for z exterior to D. This completes the proof.

The formula (2.23) is, in fact, valid for all $\xi(z)$ corresponding to solutions x of $Q * x = f$. Indeed we have

Theorem 6. Let x be any solution of $Q * x = f$, so that $\widehat{Q(z)\xi(z)} = \phi(z)$, where $\phi(z)$ is analytic for $|z| > \gamma^+$ and $|z| < \gamma^-$ and $Q(z)$ is analytic for $\rho^- < |z| < \rho^+$, with the inequalities (2.5) applying. Let D and C be selected so that (2.16), (2.18) are valid, with no singularity of $Q(z)$ occurring on D. Then the formula (2.23) remains valid with $\widetilde{Q(z)\xi(z)}$ denoting the internal part of $Q(z)\xi(z)$ relative to C.

Proof. From Corollary 4 we see that $\xi(z)$ is also analytic for $|z| > \gamma^+$ and for $|z| < \gamma^-$.

Proof. For w in the exterior of C, the external part of $Q(w)\xi(w)$ is

$$\widehat{Q(w)\xi(w)} = \frac{1}{2\pi i}\int_C \frac{Q(z)\xi(z)dz}{w - z} = \text{(from (2.21))}$$

$$= -\frac{1}{4\pi^2}\int_C \frac{Q(z)}{w - z}\int_D Q(\zeta)^{-1}\left(\frac{\phi(\zeta) + \psi(\zeta)}{z - \zeta}\right)d\zeta dz$$

Now

$$-\frac{1}{4\pi^2}\int_C \frac{Q(z)}{w - z}\int_D \frac{Q(\zeta)^{-1}\phi(\zeta)}{z - \zeta}\, d\zeta dz = -\frac{1}{4\pi^2}\int_D\int_C \frac{Q(z)dz}{(w - z)(z - \zeta)}\, Q(\zeta)^{-1}\phi(\zeta)d\zeta$$

$$= \frac{1}{2\pi i}\int_D\int_C Q(z)\left(\frac{1}{w - z} + \frac{1}{z - \zeta}\right)dz\, \frac{Q(\zeta)^{-1}\phi(\zeta)}{w - \zeta}\, d\zeta = \phi(w)$$

because $\frac{1}{w - z}$ and $Q(z)$ are holomorphic in the interior of z,

$$\frac{1}{2\pi i}\int_C \frac{Q(z)dz}{z - \zeta} = Q(\zeta) ,$$

and ϕ is external relative to D. The same process gives

$$-\frac{1}{4\pi^2}\int_C \frac{Q(z)}{w - z}\int_D \frac{Q(\zeta)^{-1}\psi(\zeta)}{z - \zeta}\, d\zeta dz = \frac{1}{2\pi i}\int_D \frac{\psi(\zeta)}{w - \zeta}\, d\zeta = \frac{1}{2\pi i}\int_C \frac{\psi(z)}{w - z}\, dx = 0 \quad (2.24)$$

because $\frac{1}{w - z}$ is holomorphic in the interior of C and ψ is internal relative to C. Thus

$$\widehat{Q(w)\xi(w)} = \phi(w)$$

and this implies $Q * x = f$ as required.

Now let w lie in the interior of D. Since ψ is internal relative to C and ϕ is external relative to D

$$\psi(w) = \frac{1}{2\pi i}\int_D \frac{\psi(\zeta)}{\zeta - w}\, d\zeta = \frac{1}{2\pi i}\int_D \frac{(\psi(\zeta) + \phi(\zeta))d\zeta}{\zeta - w} . \quad (2.25)$$

Now, using the formula (2.21) for $\xi(z)$, still for w in the interior of D,

$$
\begin{aligned}
Q(w)\xi(w) &= -\frac{1}{4\pi^2}\int_C \frac{Q(z)}{z-w}\int_D \frac{Q(\zeta)^{-1}(\psi(\zeta)+\phi(\zeta))d\zeta}{z-\zeta}\,dz \\
&= -\frac{1}{4\pi^2}\int_D\int_C Q(z)\Big(\frac{1}{z-\zeta}-\frac{1}{z-w}\Big)dz\,\frac{Q(\zeta)^{-1}(\psi(\zeta)+\phi(\zeta))}{z-w}\,d\zeta \\
&= \frac{1}{2\pi i}\int_D \frac{(\psi(\zeta)+\phi(\zeta))d\zeta}{\zeta-w} - Q(w)\,\frac{1}{2\pi i}\int_D \frac{Q(\zeta)^{-1}(\psi(\zeta)+\phi(\zeta))d\zeta}{\zeta-w}\,. \qquad (2.26)
\end{aligned}
$$

Combining (2.25) with (2.26) we have

$$Q(w)^{-1}(\psi(w) - \widehat{Q(w)\xi(w)}) = \frac{1}{2\pi i}\int_D \frac{Q(\zeta)^{-1}(\psi(\zeta)+\phi(\zeta))d\zeta}{\zeta-w}$$

from which we conclude that (2.22) is internal relative to D, as required for the theorem. Then the formula (2.21) for $\xi(z)$ immediately gives the result (2.23), for, if $Q(\zeta)^{-1}(\psi(\zeta) - \widehat{Q(\zeta)\xi(\zeta)})$ is internal relative to D,

$$\frac{1}{2\pi i}\int_D \frac{Q(\zeta)^{-1}(\psi(\zeta) - \widehat{Q(z)\xi(\zeta)})d\zeta}{z-\zeta} = 0$$

for z exterior to D. This completes the proof.

The formula (2.23) is, in fact, valid for all $\xi(z)$ corresponding to solutions x of $Q * x = f$. Indeed we have

Theorem 6. <u>Let x be any solution of $Q * x = f$, so that $\widehat{Q(z)\xi(z)} = \phi(z)$, where $\phi(z)$ is analytic for $|z| > \gamma^+$ and $|z| < \gamma^-$ and $Q(z)$ is analytic for $\rho^- < |z| < \rho^+$, with the inequalities (2.5) applying. Let D and C be selected so that (2.16), (2.18) are valid, with no singularity of $Q(z)$ occurring on D. Then the formula (2.23) remains valid with $\widehat{Q(z)\xi(z)}$ denoting the internal part of $Q(z)\xi(z)$ relative to C.</u>

<u>Proof</u>. From Corollary 4 we see that $\xi(z)$ is also analytic for $|z| > \gamma^+$ and for $|z| < \gamma^-$.

<u>Proof</u>. For w in the exterior of C, the external part of $Q(w)\xi(w)$ is

$$\widehat{Q(w)\xi(w)} = \frac{1}{2\pi i}\int_C \frac{Q(z)\xi(z)dz}{w - z} = \text{(from (2.21))}$$

$$= -\frac{1}{4\pi^2}\int_C \frac{Q(z)}{w - z}\int_D Q(\zeta)^{-1}\left(\frac{\phi(\zeta) + \psi(\zeta)}{z - \zeta}\right)d\zeta dz$$

Now

$$-\frac{1}{4\pi^2}\int_C \frac{Q(z)}{w - z}\int_D \frac{Q(\zeta)^{-1}\phi(\zeta)}{z - \zeta}\, d\zeta dz = -\frac{1}{4\pi^2}\int_D\int_C \frac{Q(z)dz}{(w - z)(z - \zeta)}\, Q(\zeta)^{-1}\phi(\zeta)d\zeta$$

$$= \frac{1}{2\pi i}\int_D\int_C Q(z)\left(\frac{1}{w - z} + \frac{1}{z - \zeta}\right)dz\, \frac{Q(\zeta)^{-1}\phi(\zeta)}{w - \zeta}\, d\zeta = \phi(w)$$

because $\frac{1}{w - z}$ and $Q(z)$ are holomorphic in the interior of z,

$$\frac{1}{2\pi i}\int_C \frac{Q(z)dz}{z - \zeta} = Q(\zeta) \ ,$$

and ϕ is external relative to D. The same process gives

$$-\frac{1}{4\pi^2}\int_C \frac{Q(z)}{w - z}\int_D \frac{Q(\zeta)^{-1}\psi(\zeta)}{z - \zeta}\, d\zeta dz = \frac{1}{2\pi i}\int_D \frac{\psi(\zeta)}{w - \zeta}\, d\zeta = \frac{1}{2\pi i}\int_C \frac{\psi(z)}{w - z}\, dx = 0 \quad (2.24)$$

because $\frac{1}{w - z}$ is holomorphic in the interior of C and ψ is internal relative to C.

Thus

$$\widehat{Q(w)\xi(w)} = \phi(w)$$

and this implies $Q * x = f$ as required.

Now let w lie in the interior of D. Since ψ is internal relative to C and ϕ is external relative to D

$$\psi(w) = \frac{1}{2\pi i}\int_D \frac{\psi(\zeta)}{\zeta - w}\, d\zeta = \frac{1}{2\pi i}\int_D \frac{(\psi(\zeta) + \phi(\zeta))d\zeta}{\zeta - w}\ . \quad (2.25)$$

$$y_\ell^- = \sum_{k=-\infty}^{-\ell} Q_k x_{k+\ell}$$

and then

$$y_\ell = y_\ell^+ + y_\ell^- = \sum_{j=-\infty}^{\infty} Q_k x_{k+\ell} = (Q * x)_\ell = \phi_\ell \ .$$

Then since $\phi(z)$ and $\eta(z)$, given by (2.17), are both external functions relative to D, (2.15) follows and the proof is complete.

Proposition 3 tells us, given Q and x, a fairly elegant way to compute $f = Q * x$. Now let us reverse the process. Let us suppose that the sequence $f = \{f_k \in X | -\infty < k < \infty\}$ has z-transform $\phi(z)$ external relative to D. What may we then say about those sequences $x = \{x_k \in X | -\infty < k < \infty\}$ such that $Q * x = f$? We have

Theorem 5. Assume $Q(z)$ has isolated singularities. Let C lie in the extension of D with no singularity of $Q(z)$ lying on D. Let f have z-transform $\phi(z)$ external relative to D. Let $Q(z)$, the discrete Fourier transform of $\{Q_k \in L(X,X) | -\infty < k < \infty\}$ be analytic for $\rho^- < |z| < \rho^+$ and let the radii of C^+, C^-, D^+, D^- satisfy (2.16), (2.18). Let $\psi(z)$ be an arbitrary function interval relative to C. Then the sequence $\{x_k \in X | -\infty < k < \infty\}$ with z-transform

$$\xi(z) = \frac{1}{2\pi i} \int_D Q(\zeta)^{-1} \left(\frac{\phi(\zeta) + \psi(\zeta)}{z - \zeta}\right) d\zeta \ , \tag{2.21}$$

defined for z exterior to D, satisfies

$$Q * x = f \ .$$

Moreover, the function

$$Q(z)^{-1}(\psi(z) - \widetilde{Q(z)\xi(z)}) \tag{2.22}$$

is internal relative to D and thus, also,

$$\xi(z) = \frac{1}{2\pi i} \int_D Q(\zeta)^{-1} \left(\frac{\phi(\zeta) + \widetilde{Q(\zeta)\xi(\zeta)}}{z - \zeta}\right) d\zeta \ , \tag{2.23}$$

where $\widetilde{Q(z)\xi(x)}$ denotes the internal part of $Q(z)\xi(z)$ relative to C.

Proof of Proposition 3. We will compute the coefficients y_ℓ, $-\infty < \ell < \infty$, of the expansion of the function

$$\eta(z) = \widehat{Q(z)\xi(z)} , \tag{2.17}$$

computed relative to D. To this end, let $C = C^+ - C^-$ be such that the radii, c^+ and c^-, of C^+ and C^-, respectively, satisfy

$$\rho^+ > c^+ > d^+, \qquad \rho^- < c^- < d^- . \tag{2.18}$$

Then

$$y_\ell = \frac{1}{2\pi i}\int_C \eta(z) z^{\ell-1} dz = -\frac{1}{4\pi^2}\int_C\int_D \frac{Q(\zeta)\xi(\zeta)}{z-\zeta}\, d\zeta z^{\ell-1} dz$$

$$= -\frac{1}{4\pi^2}\int_D\int_C Q(\zeta)\xi(\zeta)\frac{z^{\ell-1}}{z-\zeta}\, dz d\zeta .$$

For $z \in C$ and $\zeta \in D$

$$\frac{z^{\ell-1}}{z-\zeta} = z^{\ell-2} + \zeta z^{\ell-3} + \zeta^2 z^{\ell-4} + \ldots$$

so that

$$y_\ell = -\frac{1}{4\pi^2}\int_D \sum_{j=0}^{\infty} \zeta^j \int_C z^{\ell-j-2} dz Q(\zeta)\xi(\zeta)d\zeta = \frac{1}{2\pi i}\int_D \zeta^{\ell-1} Q(\zeta)\xi(\zeta)d\zeta$$

$$= \frac{1}{2\pi i}\int_{D^+} \zeta^{\ell-1} Q(\zeta)\xi(\zeta)d\zeta - \frac{1}{2\pi i}\int_{D^-} \zeta^{\ell-1} Q(\zeta)\xi(\zeta)d\zeta \equiv y_\ell^+ + y_\ell^- . \tag{2.19}$$

On D^+, letting $m = j - k$,

$$Q(\zeta)\xi(\zeta) = \Big(\sum_{k=-\infty}^{\infty} Q_j\zeta^k\Big)\Big(\sum_{k=1}^{\infty} x_j\zeta^{-j}\Big) = \sum_{m=-\infty}^{\infty}\Big(\sum_{k=1-m}^{\infty} Q_k x_{k+m}\Big)\zeta^{-m} . \tag{2.20}$$

Substituting (2.20) into the integral for y_ℓ^+ in (2.19) we have

$$y_\ell^+ = \frac{1}{2\pi i}\sum_{m=-\infty}^{\infty}\Big(\sum_{k=1-m}^{\infty} Q_k x_{k+m}\Big)\int_{D^+}\zeta^{\ell-m-1}d\zeta = \sum_{k=1-\ell}^{\infty} Q_k x_{k+\ell} .$$

A similar computation on D- gives

z exterior to D, by

$$h_k = \frac{1}{2\pi i} \int_D h(z) z^{k-1} dz \ .$$

Similar considerations apply to functions internal relative to D.

<u>**Proposition 3**. Let</u> x, Q, <u>as described at the beginning of this section, have transforms</u> $\xi(z)$, $Q(z)$ <u>and let</u> $\phi(z)$ <u>be the discrete Laplace transform of</u> $Q * x$. <u>Then</u>

$$\phi(z) = \widehat{Q(z)\xi(z)} \ , \tag{2.15}$$

<u>relative to any contour</u> $D = D^+ - D^-$ <u>with</u>

$$\rho^+ > d^+ > \gamma^+, \quad \rho^- < d^- < \gamma^- \ . \tag{2.16}$$

<u>**Remark**</u>. In other words, the "z-transform", $\phi(z)$, of the convolution $Q * x$ is the external function, relative to D as described, corresponding to the ordinary product $Q(z)\xi(z)$ of the discrete Fourier transform of $Q = \{Q_k\}$ and the discrete Laplace transform, or "z-transform", $\xi(z)$, of the sequence $x = \{x_k\}$.

<u>**Corollary 4**. For each</u> $\varepsilon > 0$ <u>we can find</u> $M + (\varepsilon)$, $M - (\varepsilon)$ <u>such that</u>

$$\| f_\ell \| = \| Q * x)_\ell \| \begin{cases} < M^+(\varepsilon)(\gamma^+ + \varepsilon)^\ell, & \ell = 1,2,3,\ldots \\ < M^-(\varepsilon)(\gamma^- - \varepsilon)^\ell, & \ell = 0,-1,-2,-3,\ldots \ . \end{cases}$$

<u>**Proof**</u>. Let $\varepsilon > 0$ be selected and let D, as in Proposition 3, be chosen so that $\gamma^+ + \varepsilon > d^+ > \gamma^+$, $\gamma^- - \varepsilon < d^- < \gamma^-$. Let $C = C^+ - C^-$ with

$$c^+ = \gamma^+ + \varepsilon, \qquad c^- = \gamma^- - \varepsilon \ .$$

Then (2.9), (2.10) give

$$f_\ell = \frac{1}{2\pi i} \int_C \phi(z) z^{\ell-1} dz = -\frac{1}{4\pi^2} \int_C z^{\ell-1} \int_D \frac{Q(w)\xi(w)}{z - w} \, dw dz \ .$$

Since the integral over C^- vanishes for $\ell = 1,2,3,\ldots$ and the integral over C^+ vanishes for $\ell = 0,-1,-2,-3,\ldots$, the estimates follow immediately.

D, then as $\varepsilon \to 0$ the functions

$$\hat{h}\big|_{\hat{D}_\varepsilon}, \quad \tilde{h}\big|_{\tilde{D}_\varepsilon}$$

converge in the $L^2[0,2\pi)$ norm $(z = |z|e^{i\theta},\ 0 \le \theta < 2\pi)$ to functions $\hat{h}$, $\tilde{h}$ in that space (we may also write $\hat{h}, \tilde{h} \in L^2(D)$ unambiguously) such that

$$h = \hat{h} + \tilde{h} . \tag{2.12}$$

We refer to $\hat{h}\big|_D$ and $\tilde{h}\big|_D$ as the external and internal parts of h, respectively. This defines a decomposition of $L^2(D)$:

$$L^2(D) = \hat{L}^2(D) + \tilde{L}^2(D) \tag{2.13}$$

and the (non-orthogonal, in general) projections - they are readily seen to be that - $\hat{P}$, $\tilde{P}$ defined by

$$\hat{P}(h) = \hat{h}, \quad \tilde{P}(h) = \tilde{h} ,$$

mapping $L^2(D)$ onto $\hat{L}^2(D)$, $\tilde{L}^2(D)$, respectively, may be shown to be bounded and, clearly,

$$\hat{P} + \tilde{P} = I \ \text{ on } \ L^2(D) . \tag{2.14}$$

If h is analytic in the exterior of D with square integrable limiting values on D and if $h = \hat{h}$ relative to D, then we will say that h is an external function, or, simply, that h is external, relative to D. Internal functions are defined in a similar manner. Thus for external functions

$$h(z) = (\hat{P}h)(z) = \frac{1}{2\pi i}\int_D \frac{h(\zeta)d\zeta}{z - \zeta}$$

for z exterior to D while for internal functions

$$h(z) = (\tilde{P}h)(z) = \frac{1}{2\pi i}\int_D \frac{h(\zeta)d\zeta}{\zeta - z}$$

for z in the interior of D. If C lies exterior to D and h is exterior relative to D, then it is also exterior relative to C. If h is exterior relative to C and analytic in the exterior of D, then h is also exterior relative to D. If h is external relative to D then $h(z)$ is the z-transform of the sequence h_k defined, for

$$\widehat{Q(w)\xi_0(w)} = \frac{1}{2\pi i}\int_C \frac{Q(z)\xi_0(z)dz}{w-z}$$

$$= -\frac{1}{4\pi^2}\int_{C^+}\frac{1}{w-z}\int_{D^-}\frac{Q(\zeta)\xi^-(\zeta)d\zeta}{z-\zeta}\,dz - \frac{1}{2\pi i}\int_{C^-}\frac{Q(z)\xi^-(z)dz}{w-z}\,.$$

The first integral on the right hand of (3.3) is, for $|w| > c^+$,

$$-\frac{1}{4\pi^2}\int_{D^-} Q(\zeta)\xi^-(\zeta)\int_{C^+}\frac{dz}{(w-z)(z-\zeta)}\,d\zeta$$

$$= -\frac{1}{4\pi^2}\int_{D^-}\frac{Q(\zeta)\xi^-(\zeta)}{w-\zeta}\int_{C^+}\left(\frac{1}{w-z}+\frac{1}{z-\zeta}\right)dz\,d\zeta = \frac{1}{2\pi i}\int_{D^-}\frac{Q(\zeta)\xi^-(\zeta)d\zeta}{w-\zeta}$$

and therefore, still for $|w| > c^+$,

$$\widehat{Q(w)\xi_0(w)} = \frac{1}{2\pi i}\int_{D^-}\frac{Q(\zeta)\xi^-(\zeta)d\zeta}{w-\zeta} - \frac{1}{2\pi i}\int_{C^-}\frac{Q(z)\xi^-(z)dz}{w-z} = 0$$

since the integrand $Q(z)\xi^-(z)/w-z$ is holomorphic on and between D^- and C^-. It follows that $(\widehat{Q(w)\xi(w)})^+ = 0$, i.e., that

$$(Q * \xi_0)_k = 0, \qquad k = 1,2,3,\ldots\,,$$

and we conclude that x_0, of which $\xi_0(z)$ is the z-transform, is a unilateral solution of the homogeneous equation.

Now if $f_k = 0$, $k = 0,-1,-2,\ldots$, then its z transform is, for $|z| > \gamma^+$,

$$\phi(z) = \phi^+(z) = \sum_{k=1}^{\infty} f_k z^{-k}\,,$$

and $\phi(z) = \phi^-(z)$ vanishes identically for $|z| < \gamma^-$. Taking D^+ to have radius d^+ large enough so that no singularities of $Q(z)$ lie in the set $|z| > d^+$ and selecting d^- so that no singularities of $Q(z)$ are on D^-, in the formula

$$\xi_\phi(z) = \frac{1}{2\pi i}\int_D \frac{Q(\zeta)^{-1}\phi(\zeta)d\zeta}{z-\zeta}$$

the integral over D^- vanishes because $\phi(\zeta) = 0$ there and thus

$$\xi_\phi(z) = \frac{1}{2\pi i}\int_{D^+} \frac{Q(\zeta)^{-1}\phi(\zeta)d\zeta}{z-\zeta} = \frac{1}{2\pi i}\int_{D^+} \frac{Q(\zeta)^{-1}\phi^+(\zeta)d\zeta}{z-\zeta}$$

for all z exterior to D. For $|z| < d^-$ we must have $\xi_\phi(z) \equiv 0$ because $Q(\zeta)^{-1}\phi^+(\zeta)/(z-\zeta)$ is holomorphic for $|z| > d^+$, and we conclude, applying Theorem 5, that $\xi_\phi(z)$ is the z-transform of a sequence x_ϕ such that $x_\phi^- = \{0\}$ and $Q * x = f$. Then $x = x_0 + x_\phi$ must be the unique unilateral solution of $Q * x = f$ with the given initial history x^- and, clearly, $\xi^+(z) = \xi_0^+(z) + \xi_\phi^+(z)$ satisfies (3.2).

The foregoing development of a transform theory for $Q * x = f$ contains, we believe, notions and results which may prove interesting and useful in a number of different connections. For us here, the main point is to be able to supply the

Proof of Theorem 2. We start with the formula (3.2) which, specialized to unilaterally homogeneous solutions, gives, with D in the statement of Theorem 2 taking the place of D^-,

$$\xi^+(z) = \frac{1}{2\pi i} Q(z)^{-1} \int_D \frac{Q(\zeta)\xi^-(\zeta)d\zeta}{z-\zeta} \equiv Q(z)^{-1}q(z) ,$$

the radius, d, of D satisfying $\rho^- < d < \gamma^-$.

Next we let C be a circular contour of radius $c > d$, positively oriented and centered at $z = 0$. Since Q_0 is nonsingular, we may suppose c chosen large enough so that all singular points of $Q(z)$ lie in the region $|z| < c$. Then, from (2.9), the coefficient vectors x_k, $k = 1,2,3,\ldots$, of x^+ are given by

$$x_k = \frac{1}{2\pi i}\int_C \xi^+(z)z^{k-1}dz$$

$$= \frac{1}{2\pi i}\int_{C-S} \xi^+(z)z^{k-1}dz + \frac{1}{2\pi i}\int_S \xi^+(z)z^{k-1}dz \equiv x_{k,F} + x_{k,\sigma} .$$

Then, in view of the formula (3.4), we have (1.13) with $q(z)$ as described in (1.14). The estimate (1.15) is then immediate since $|z| = \sigma$ on S and $|\xi^{+}(z)|$ is bounded on S. The formula (1.16) is an immediate consequence of the calculus of residues, of course.

Let ζ be a singular point of $Q(z)$ lying between C and S. We assume that $Q(z)^{-1}$ has a pole of order ν at the point ζ so that, with $\hat{Q}_0(z)$ holomorphic near ζ

$$Q(z)^{-1} = \hat{Q}_0(z) + \frac{\hat{Q}_{-1}}{z - \zeta} + \dots + \frac{\hat{Q}_{-\nu}}{(z - \zeta)^{\nu}} .$$

If we let Γ_{ζ} be a small circle, centered at ζ and containing no other singular points of $Q(z)$ in its interior or on Γ_{ζ} itself, then the residue of $z^{k-1}\xi^{+}(z) = z^{k-1}Q(z)^{-1}q(z)$ at $z = \zeta$ is

$$r_{\zeta} = \frac{1}{2\pi i}\int_{\Gamma_{\zeta}} z^{k-1}Q(z)^{-1}q(z)dz$$

$$= \frac{1}{2\pi i}\int_{\Gamma_{\zeta}} z^{k-1}[\hat{Q}_0(z) + \frac{\hat{Q}_{-1}}{z - \zeta} + \dots + \frac{\hat{Q}_{-\nu}}{(z - \zeta)^{\nu}}]q(z)dz$$

$$= \frac{1}{2\pi i}\int_{\Gamma_{\zeta}} z^{k-1}[\frac{\hat{Q}_{-1}}{z - \zeta} + \dots + \frac{\hat{Q}_{-\nu}}{(z - \zeta)^{\nu}}]q(z)dz$$

$$= \zeta^{k-1}\hat{Q}_{-1}q(\zeta) + \frac{1}{2\pi i}\int_{\Gamma_{\zeta}} z^{k-1}[\frac{\hat{Q}_{-2}}{(z - \zeta)^{2}} + \dots + \frac{\hat{Q}_{-\nu}}{(z - \zeta)^{\nu}}]q(z)dz$$

$$= \zeta^{k-1}q(\zeta) + \frac{1}{2\pi i}\int_{\Gamma_{\zeta}} [\frac{\hat{Q}_{-2}}{(z - \zeta)} + \dots + \frac{\hat{Q}_{-\nu}}{(\nu - 1)(z - \zeta)^{\nu-1}}]\frac{d}{dz}(z^{k-1}q(z))dz$$

$$= \zeta^{k-1}\hat{Q}_{-1}q(\zeta) + \hat{Q}_{-2}\frac{d}{dz}(z^{k-1}q(z))\Big|_{z=\zeta}$$

$$+ \frac{1}{2\pi i}\int_{\Gamma_{\zeta}} [\frac{\hat{Q}_{-3}}{2(z - \zeta)} + \dots + \frac{\hat{Q}_{-\nu}}{(\nu - 1)(\nu - 2)(z - \zeta)^{\nu-2}}]\frac{d^2}{dz^2}(z^{k-1}q(z))dz$$

$$= \hat{Q}_{-1}\zeta^{k-1}q(\zeta) + \hat{Q}_{-2}((k - 1)\zeta^{k-2}q(\zeta) + \zeta^{k-1}q^{\nu}(\zeta))$$

$$+ \dots + \frac{\hat{Q}_{-\nu}}{(\nu - 1)!}((k - 1)\dots(k - \nu + 1)\zeta^{k-2}q(\zeta) + \dots + \zeta^{k-1}q^{(\nu-1)}(\zeta))$$

$$= \zeta^{k-1}\left[\hat{Q}_{-1}q(\zeta) + \hat{Q}_{-2}q^{\nabla}(\zeta) + \cdots + \frac{\hat{Q}_{-\nu}}{(\nu-1)!}q^{(\nu-1)}(\zeta)\right]$$

$$+ (k-1)\zeta^{k-2}\left[\hat{Q}_{-2}q(\zeta) + \cdots + \frac{\hat{Q}_{-\nu}}{(\nu-1)!}q^{(\nu-2)}(\zeta)\right]$$

$$+ \cdots + (k-1)\ldots(k-\nu+1)\zeta^{k-\nu}\frac{\hat{Q}_{-\nu}}{(\nu-1)!}q(\zeta) \ . \tag{3.5}$$

From this it is clear that the solution sequence $x_{k,F}$ is a linear combination of solutions of the form (1.17). It remains only to show that the vectors

$$p_j^{\ell} = \hat{Q}_{-j}q^{(\ell)}(\zeta), \quad j = 1,2,\ldots,\nu, \quad \ell = 0,\ldots,\nu-1$$

agree with the description (1.18), of the vectors p appearing in (1.17). A brief inspection will show that this is the case, provided we can show that

$$Q_k\hat{Q}_{-j} = 0, \quad k = 0,\ldots,j-1, \quad j = 1,2,\ldots,\nu \ .$$

To this end we observe that since

$$Q(z)Q(z)^{-1} = I \ ,$$

$$(Q(z)Q(z)^{-1})^{\nabla} = Q^{\nabla}(z)Q(z)^{-1} + Q(z)(Q(z)^{-1})^{\nabla} = 0$$

$$(Q(z)Q(z)^{-1})^{\nabla\nabla} = Q(z)^{\nabla\nabla}Q(z)^{-1} + 2Q^{\nabla}(z)(Q(z)^{-1})^{\nabla} + Q(z)(Q(z)^{-1})^{\nabla\nabla} = 0$$

etc., etc., the negative indexed terms in the Laurent expansions must be zero. Starting in each case with the coefficient of the most negative power of z, we find that the coefficients are, respectively as the equation is shown above:

$$Q_0\hat{Q}_{-\nu},\ Q_1\hat{Q}_{-\nu} + Q_0\hat{Q}_{-\nu+1}, Q_2\hat{Q}_{-\nu} + Q_1\hat{Q}_{-\nu+1} + Q_0\hat{Q}_{-\nu+2},\ldots,$$

$$Q_0\hat{Q}_{-\nu}, Q_1\hat{Q}_{-\nu} + (-\nu+1)Q_0\hat{Q}_{-\nu+1},\ 2Q_2\hat{Q}_{-\nu} + (-\nu+1)Q_1\hat{Q}_{-\nu+1} + (-\nu+2)Q_0\hat{Q}_{-\nu+2},\ldots,$$

$$Q_0\hat{Q}_{-\nu},\ 2Q_2\hat{Q}_{-\nu} + 2(-\nu+1)Q_1\hat{Q}_{-\nu+1} + (-\nu+1)(-\nu+2)Q_0\hat{Q}_{-\nu+2},\ldots,$$

etc., etc.

and from this it may be seen that

$$Q_0\hat{Q}_{-\nu} = 0,\ Q_0\hat{Q}_{-\nu+1} = 0,\ Q_0\hat{Q}_{-\nu+2} = 0,\ldots,Q_0\hat{Q}_{-1} = 0$$

$$Q_1\hat{Q}_{-\nu} = 0,\ Q_1\hat{Q}_{-\nu+2} = 0,\ldots,Q_1\hat{Q}_{-2} = 0$$

$$Q_2\hat{Q}_{-\nu} = 0,\ Q_2\hat{Q}_{-\nu+1},\ldots,Q_2\hat{Q}_{-3} = 0 \ \ldots$$

and since $Q_k = Q^{(k)}(\zeta)/k!$ we have the desired result. With this the proof of Theorem 2 is complete and, with it as a tool, we may now turn to the proof of Theorem 1 which was the original motivation of this paper.

4. PROOF OF THEOREM 1

With $x(t)$, $W(t,s)$ as in the statement of the theorem, for $t > 0$ we have

$$x(t) = \int_{-\infty}^{t} W(t,s)x(s)ds = \int_{(j-1)T}^{t} W(t,s)x(s)ds + \int_{-\infty}^{(j-1)T} W(t,s)x(s)ds$$

where j is the largest integer such that $(j-1)T < t$. For $\ell = 1,2,3,\ldots$, we define

$$x_\ell(\tau) = x((\ell-1)T + \tau), \quad \tau \in [0,T] . \tag{4.1}$$

Then, with $t = (j-1)T + \tau$, $s = (\ell-1)T + \sigma$, $\tau,\sigma \in [0,T]$,

$$\begin{aligned} x_j(\tau) &- \int_0^\tau W((j-1)T + \tau,(j-1)T + \sigma)x_j(\sigma)d\sigma \\ &- \sum_{\ell=-\infty}^{j-1} \int_0^T W((j-1)T + \tau,(\ell-1)T + \sigma)x_\ell(\sigma)d\sigma = 0 \end{aligned} \tag{4.2}$$

Using the periodicity relation (1.4), we have

$$W((j-1)T + \tau,(\ell-1)T + \sigma) = W(\tau,(\ell-j)T + \sigma) \equiv W_{\ell-j}(\tau,\sigma)$$

and (4.2) becomes

$$x_j(\tau) - \int_0^\tau W_0(\tau,\sigma)x_j(\sigma)d\sigma - \sum_{k=-\infty}^{-1} \int_0^T W_k(\tau,\sigma)x_{k+j}(\sigma)d\sigma = 0 . \tag{4.3}$$

If we now define the operators P_0, P_k, $k = -1,-2,-3,\ldots$, for $x \in L_m^2[0,T]$, by

$$(P_0x)(\tau) = x(\tau) - \int_0^\tau W_0(\tau,\sigma)x(\sigma)d\sigma ,$$

$$(P_kx)(\tau) = - \int_0^T W_k(\tau,\sigma)x(\sigma)d\sigma ,$$

it is classical ([5]) that P_0 is bounded and boundedly invertible while the operators P_k, $k = -1,-2,-3,\ldots$, are all compact. Multiplying by $(P_0)^{-1}$, we see that, with $Q_k = (P_0)^{-1}P_k$, $Q_0 = I$,

$$\sum_{k=-\infty}^{0} Q_k x_{k+j} \; (= (Q * x)_j) = 0, \quad j = 1,2,3,\ldots \tag{4.4}$$

is the expression of (4.3) as a linear recursion equation in the Hilbert space $L_m^2[0,T]$. From (1.3) it may be seen that (1.9) is true for

$$\rho > e^{-\mu T}$$

and (1.5) is true with

$$M = \Xi/\sqrt{2\gamma}$$

and

$$\gamma = e^{-\nu T}$$

Thus, if $\mu > \nu$, $\rho < \gamma$ and, as required for Theorem 2, (4.4), with Q_k, x_k defined as above, converges for $|z| > \rho$ while (1.11) converges for $|z| < \gamma$. From the definition of the operators P_k, $k = -1,-2,-3,\ldots$ and the boundedness of $(P_0)^{-1}$, the operators Q_k, $k = -1,-2,-3,\ldots$ are all compact. Therefore

$$Q(z) = I + Q_1(z)$$

where

$$Q_1(z) = \sum_{k=-\infty}^{-1} Q_k z^k ,$$

being uniformly convergent in any region $|z| > \rho + \delta$, $\delta > 0$, is compact. The theorem of Atkinson ([1],[4]) then applies to show that $Q(z)$ is "regular" in the sense described in the statement of Theorem 2. Thus Theorem 2 applies and gives the continuation sequence $x^+ = \{x_k | k = 1,2,3,\ldots\}$ in the form (1.13). With σ as in the statement of Theorem 2, we then have (1.15) for $\rho < \sigma$ i.e., with $e^{-\beta T} = \sigma$, for $\beta < \mu$, as claimed in Theorem 1.

We turn our attention then to

$$x_{k,F} = \sum_{\zeta \in Z(C,S)} \mathrm{Res}(Q(z)^{-1} q(z) z^{k-1})\Big|_{z=\zeta}$$

as already discussed in the previous section, where we have noted that the contribution to $x_{k,F}$ from each singular point ζ must take the form (3.5). Setting $e^{\lambda T} = \zeta$, it remains only to show that when the $x_{k,F} \in L_m^2[0,T]$ are taken to be the successive segments of a function $x_F(t)$, consistent with (4.1), the solution forms shown in (1.17) for the recursion equaiton correspond to solutions of (1.2) having the Floquet structure in (1.1). We have already struggled hard enough with poles of $Q(z)^{-1}$ of higher multiplicity in Section 3; let us be content here with a simple pole, leaving it to the more dedicated among us to verify the details in the other cases.

We consider, then, a solution of the form

$$x_{k,\zeta} = \zeta^{k-1} p$$

where $p \in L_m^2[0,T]$ with

$$Q(\zeta)p = 0 \ . \tag{4.5}$$

Let $\tilde{p}(\tau)$, $0 \le \rho \le T$, be the functional representation of p. Then (1.18) becomes (cf. (4.13))

$$\tilde{p}(\tau) - \int_0^\tau W_0(\tau,\sigma)\tilde{p}(\sigma)d\sigma - \sum_{j=-\infty}^{-1} \zeta^j \int_0^T W_j(\tau,\sigma)\tilde{p}(\sigma)d\sigma = 0 \ .$$

It is evident that $\tilde{p}(\tau)$ is continuous on $(0,T)$. Now

$$\begin{aligned}\tilde{p}(0) &= \sum_{j=-\infty}^{-1} \zeta^j \int_0^T W_j(0,\sigma)\tilde{p}(\sigma)d\sigma \\ &= \sum_{j=-\infty}^{-1} \zeta^j \int_0^T W(-T,(j-1)T+\sigma)\tilde{p}(\sigma)d\sigma \ ,\end{aligned}$$

while

$$\begin{aligned}\tilde{p}(T) &= \sum_{j=-\infty}^{0} \zeta^j \int_0^T W(0,(j-1)T+\sigma)\tilde{p}(\sigma)d\sigma \\ &= \zeta\Big[\sum_{j=-\infty}^{-1} \zeta^j \int_0^T W(-T,(j-1)T+\sigma)\tilde{p}(\sigma)d\sigma\Big] = \zeta\tilde{p}(0) = e^{\lambda T}\tilde{p}(0) \ .\end{aligned}$$

It follows, therefore, that if we define, for $t = (j-1)T + \tau$

$$x_\zeta(t) = \zeta^{j-1}\tilde{p}(\tau), \quad j = 1,2,3,\ldots$$

the resulting m-vector function of t is continuous with respect to t. If we now let

$$p(\tau) = e^{-\lambda\tau}\tilde{p}(\tau), \quad \tau \in [0,T] ,$$

and then let (cf. (4.1))

$$p(t) = p((j - 1)T + \tau) = p(\tau) ,$$

we have

$$p(t) = p(t + T)$$

and

$$x_\zeta(t) = \zeta^{j-1}e^{\lambda\tau}p(t) = e^{\lambda((j-1)T+\tau)}p(t) = e^{\lambda t}p(t)$$

therefore has the formed claimed in Theorem 1 for $\nu_\zeta = 1$. With this we will regard the proof of that theorem to be complete.

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We show that a functional equation in E^n:

$$x(t) = \int_{-\infty}^{t} W(t,s)x(s)ds$$

with piecewise continuous $m \times m$ matrix kernel $W(t,s)$ satisfying, for

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ABSTRACT (cont.)

T, Ω, μ > 0

$$W(t + T, s + T) = W(t,s) ,$$

$$\|W(t,s)\| \leq \Omega e^{-\mu(t-s)}, t,s \text{ real}$$

admits, for each $\beta < \mu$, a decomposition, applicable to a wide class of solutions $x(t)$ for $t \geq 0$,

$$x(t) = x_F(t) + x_\beta(t)$$

where, for some $B = B(\beta)$,

$$\|x_\beta(t)\| \leq Be^{-\beta t}, \quad t \geq 0$$

and $x_F(t)$ is a linear combination of "Floquet type" solutions

$$t^q e^{\lambda t} p(t), q(\geq 0) \in Z, \lambda \in C, Re(\lambda) > -\beta ,$$

p(t) being a continuous n-vector function such that

$$p(t + T) = p(t) .$$

The theorem is proved by converting the above equations to a convolution type linear recursion equation

$$\sum_{k=-\infty}^{0} \Omega_k x_{k+j} = 0$$

in $L_m^2[0,T]$ and studying this equation by transform methods. In the process we examine some general properties of equations

$$\sum_{k=-\infty}^{\infty} \Omega_k x_{k+j} = 0$$

within the same transform framework.

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